APPM 2360- Black Holes Project

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1 Background

The year is around 3,025,002 C.E. and this is a massive achievement for humankind. My name is Marty and I am currently alone on NASA Spaceship *Icarus*, orbiting the black hole V616 Monocerotis. I've just made a terrible error and ejected my enemy Larry out of the escape hatch. As I watch him drift towards the void, I realize that I must save him. Currently, I am orbiting at a Schwarzschild radius of 2 from the center of V616. The event horizon is at a radius of 1, so I don't have much time.

2 Initial Problem

To model the case of Larry falling towards the black hole, I present the following ordinary differential equation, where x(t) represents the radial distance from the center of the black hole to Larry:

$$\frac{dx}{dt} = e^{-x+1} - 1, \quad x(0) = 2 \tag{1}$$

where t is in seconds and x is radial distance in Schwarzschild radii. Note that equation (1) is first-order, nonlinear, autonomous, and separable. Equilibrium solutions occur when $\frac{dx}{dt} = 0$. This occurs at the event horizon of x=1 Schwarzschild radius. This makes sense because everything should be approaching the event horizon as it gets pulled toward the black hole. Due to the distortion caused by the immense gravitational field of the black hole, from my perspective, Larry will never appear to cross the event horizon of x=1. As a result, when x<1, the equation cannot depict the physics of the situation. The initial condition x(0)=2 is used, because at time t=0, Larry was on the spaceship, which is orbiting at a radius of 2.

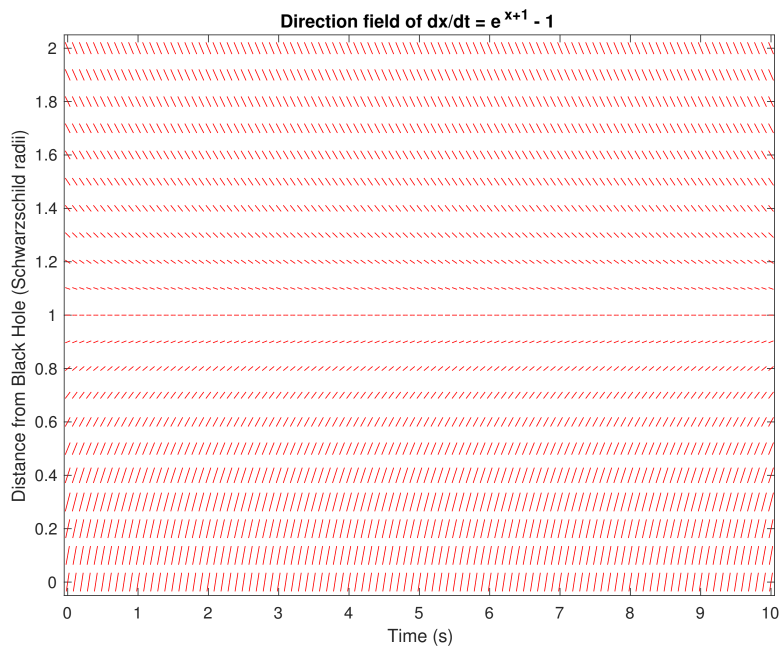

Figure 1: Direction Field of $\frac{dx}{dt} = e^{-x+1} - 1$.

Solutions for this ODE exist and are unique. By Picard's theorem, because $\frac{dx}{dt} = f(t, x)$ is continuous on the region $R = (t, x)|0 < t < 10, 1 < x < 2$ and $t_0 = 0, x_0 = 2 \in R$. Since $\frac{dx}{dt} = e^{-x+1} - 1$ is continuous for all x and t, there exists a solution for t in the interval $(t_0 - h, t_0 + h)$. Since $f_x(t, x) = -e^{-x+1}$ is also continuous for all x and all t, the solution is unique.

We can find the velocity by plugging in a given value of x(t) into the DE $v(t) = \frac{dx}{dt}$. The initial velocity of our enemy can be calculated by using x(0)=2 since this is the initial condition of our equation.

$$\frac{dx}{dt} = e^{-2+1} - 1 = e^{-1} - 1 \approx -0.6321 \text{ [Schwarzschild radii/second]}$$

The calculated analytic solution to the ODE was calculated to be:

$$x(t) = ln(e - \frac{e - e^2}{e^t})$$

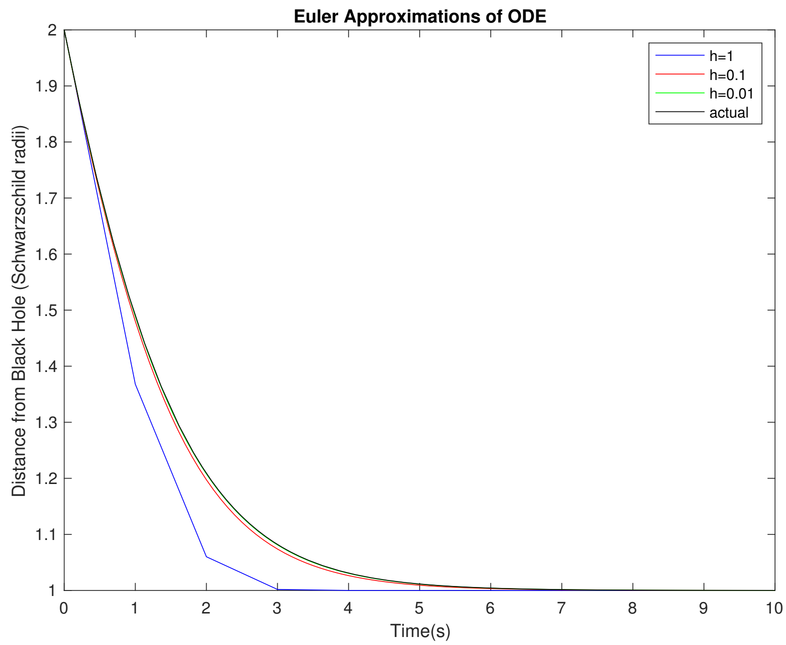

Figure 2: Euler Approximation of $\frac{dx}{dt} = e^{-x+1} - 1$ with varying step sizes.

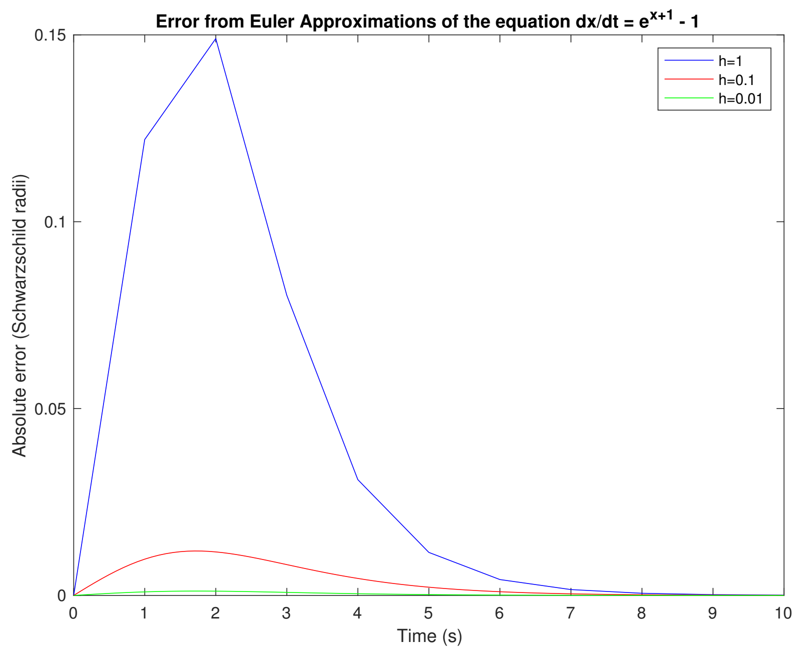

Figure 3: Absolute Error of Euler Approximations of $\frac{dx}{dt} = e^{-x+1} - 1$

The total error of each approximation is shown below.

Step Size	**Error**
1	0.4005
0.1	0.0390
0.01	0.0039

I proceeded to determine the optimal number of steps needed to optimize the computation time by plotting the total error as a function of the number of steps taken.

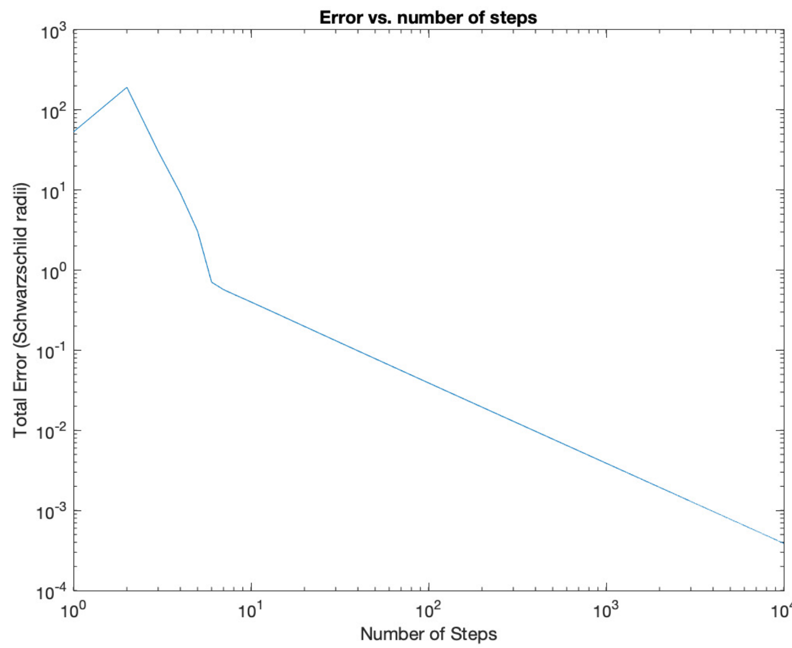

Figure 4: Log-log plot of total error as a function of steps taken N, 1¡N¡10,000

Based on the graph, we should use 6 steps, which is where the error begins decreasing linearly.

3 Problem Update

While doing my calculations, I received a transpondence from NASA that informed me that my model was not as accurate as I originally thought. The equation below is a more accurate model of the situation.

$$\frac{dx}{dt} = (\frac{1}{x(t)} - 1)\frac{1}{\sqrt{x(t)}} \tag{2}$$

Again, t is in seconds and x(t) represents radial distance in Schwarzschild radii. Note that this equation is a first-order, non-linear, autonomous, and separable but can't be solved analytically. The values for this equation are still only valid for x(t) ¿ 1 for the same reasons described above.

Equilibrium solutions occur when $\frac{dx}{dt} = 0$. This occurs for Equation (2) when x(t)=1 Schwarzschild radius. Again, this makes sense, because everything should be approaching the event horizon of the black hole. This is shown in the direction field.

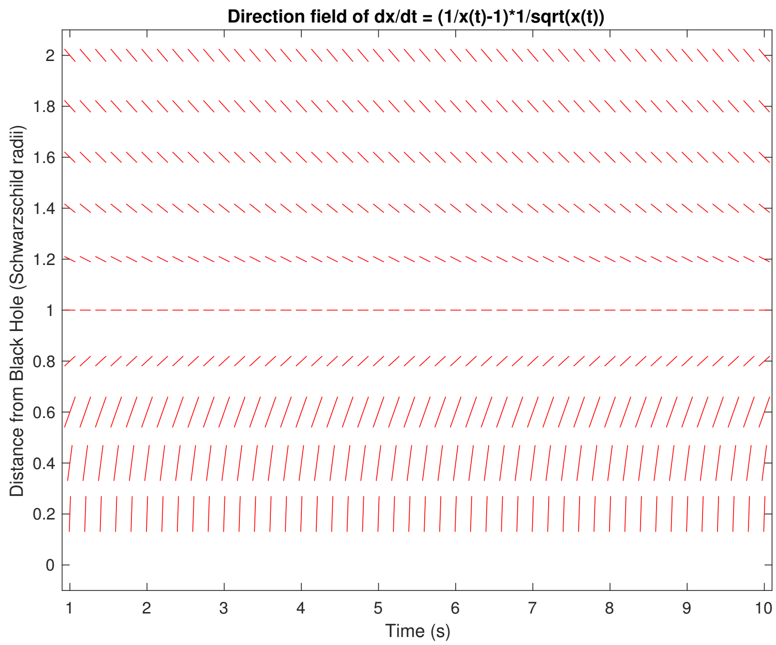

Figure 5: Direction Field of $\frac{dx}{dt} = (\frac{1}{x(t)} - 1)\frac{1}{\sqrt{x(t)}}$.

Picard's theorem does not guarantee the existence of a solution when x(t)=0. However, since our equation is only valid for x(t)¿1, this scenario is not relevant. By Picard's theorem, there exists a unique solution on the interval and initial conditions. $\frac{dx}{dt} = f(t,x)$ is continuous on the interval $R = (t,x)|0 < t < 10, 1 < x < 2$ and $t_0 = 0, x_0 = 2 \in R$ and $f_x t, x$ is continuous $R = (t,x)|0 < t < 10, 1 < x < 2$ and $t_0 = 0, x_0 = 2 \in R$ as well.

I then received the following two equations from NASA.

$$\frac{dx_e}{dt} = (\frac{1}{x_e(t)} - 1)\frac{1}{\sqrt{x_e(t)}}, \quad x_e(0) = 2 \tag{3}$$

$$\frac{dx_y}{dt} = C(\frac{1}{x_y(t)} - 1)\frac{1}{\sqrt{x_y(t)}}, \quad x_y(1) = 2 \tag{4}$$

Equation (3) represents Larry's velocity relative to the singularity of the black hole. Equation (4) represents my velocity relative to Larry's, and C represent the ratio of my velocity to Larry's.

Using these two equations, and using a timestep of 0.1 and setting my initial velocity to twice that of Larry's(C=2), I calculated the difference in our positions as a function of time.

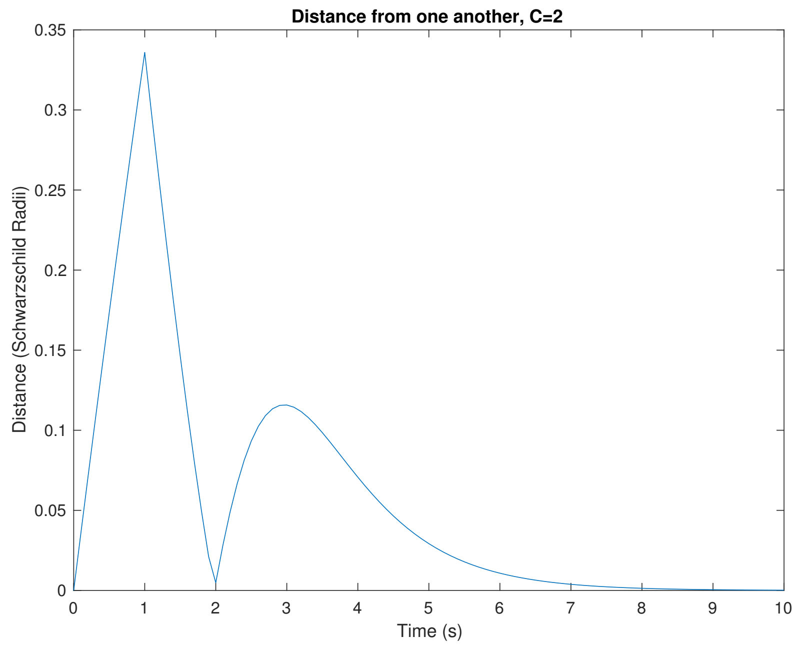

Figure 6: The graph of the difference of Larry's distance to the center of the black hole and my distance to the center of the black hole, $\frac{dx_e}{dt} - \frac{dx_y}{dt}$, C=2.

The time at which I overtake Larry is the point where the graph is equal to zero. This occurs at t=2 seconds. I determined that a Schwarzschild radius of 1.5 from center of the black hole was the closest I could get to the black hole and still make it back to the ship safely. Using the initial velocity I chose of C=2, while I would intersect Larry at 2 seconds, I wouldn't be able to reach him before I reached a Schwarzschild radius of 1.5. Therefore, I won't be able to save him if I jump at this velocity.

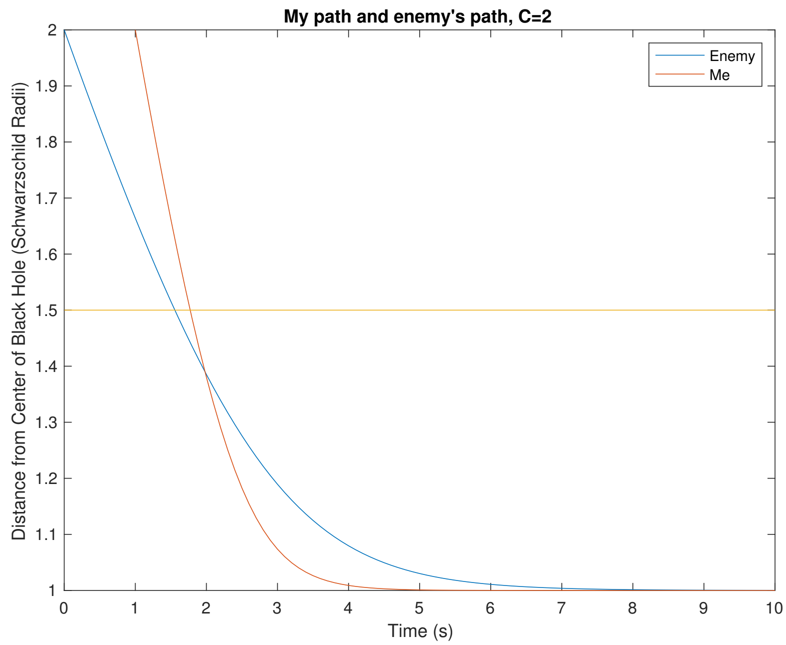

Figure 7: Larry's position and my position as a function of time, C=2

Using an initial velocity of 1.2, I determined it was still not possible to reach Larry in time. We wouldn't intersect until 5.9 seconds, but I would cross the threshold of survival of 1.5 Schwarzschild radii before then, at t=2.3 seconds. This is depicted in the graphs below.

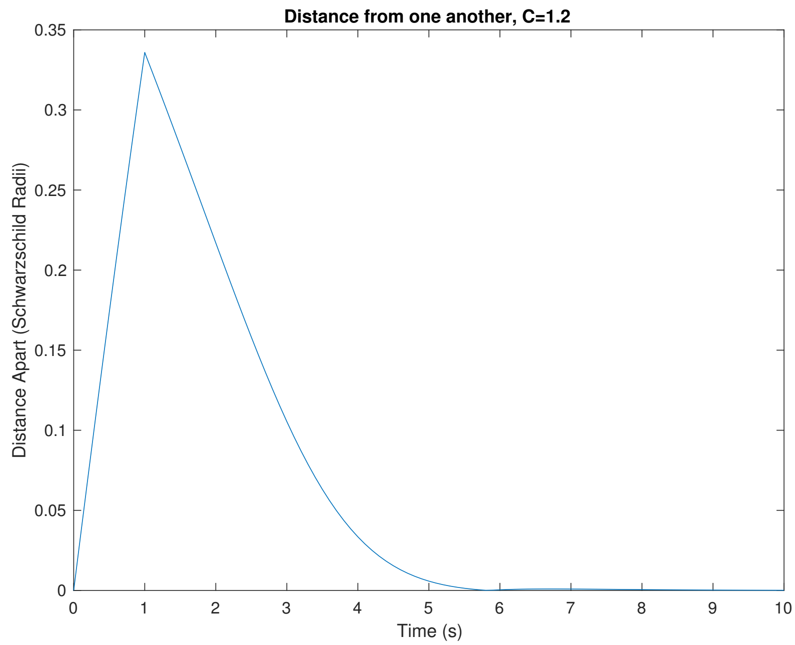

Figure 8: The graph of the difference of Larry's distance to the center of the black hole and my distance to the center of the black hole, $\frac{dx_e}{dt} - \frac{dx_y}{dt}$, C=1.2.

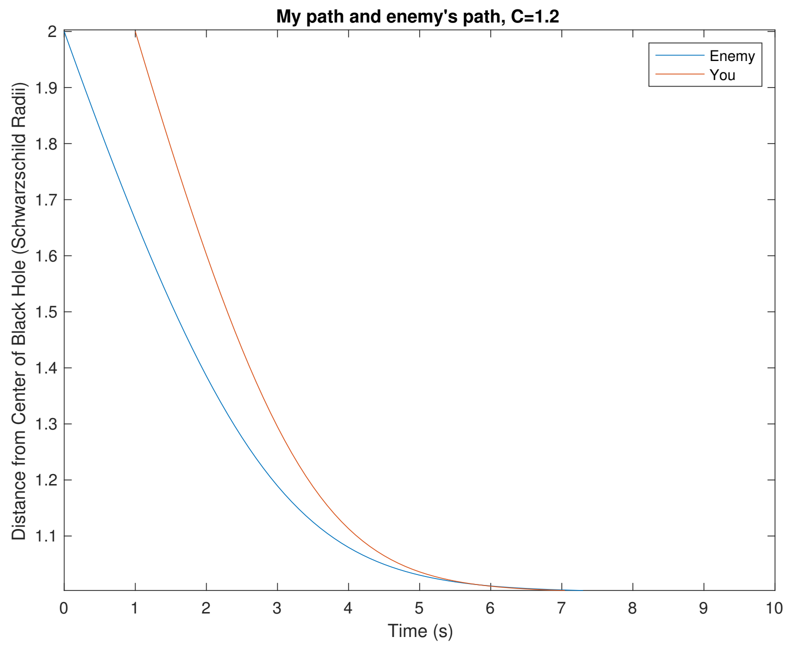

Figure 9: Larry's position and my position as a function of time, C=1.2

A summary of my results is shown below.

	Time to Overtake(s)	Time to 1.5 Schwarzschild Radii(s)
C = 1.2	5.9	2.3
C = 2	2	1.8

Thinking fast, I calculated the minimum velocity I'd need to save Larry from the black hole. I computationally iterated through values of C, beginning from 2, until there was a point that we intersected before 1.5 Schwarzschild Radii. I found that a minimum velocity of C = 2.781 is required to save Larry, which I am nowhere near.

Knowing now that it was possible, I launched out of *Icarus* using a compressed air canister to propel me with an initial velocity greater than 2.781 Schwarzschild radii per second, saving him at the last possible second.

4 Conclusion

As Larry was shoved off of *Icarus* towards the black hole, I decided to make an effort to save him. Using the initial equation(1) I calculated Larry's distance from the event horizon as a function of time analytically and by using Euler approximations of various step sizes. The smaller the step size, the more accurate the estimation was. After finding that my equations were inaccurate, I promptly calculated the minimum initial velocity required to save Larry. Eventually, I was able to save Larry from certain doom at the very last possible second.

5 Appendix

Solving Equation(1) Analytically

$$\frac{dx}{dt} = e^{-x(t)+1} - 1, \quad x(0) = 2$$
$$\frac{dx}{e^{-x(t)+1} - 1} = dt$$
$$\int \frac{dx}{e^{-x(t)+1} - 1} = \int dt$$
$$\int \frac{e^x(t)dx}{e - e^x(t)} = \int dt$$

$$u = e - e^x; du = -e^x(t)dx$$
$$-\int \frac{du}{u} = \int dt$$
$$-ln(|u|) = t + C$$

$$ln(|\frac{1}{e - e^x(t)}|) = t + C$$
$$\frac{1}{e - e^x(t)} = Ce^t$$
$$\frac{1}{Ce^t} = e - e^x(t)$$
$$e^x(t) = e - \frac{1}{Ce^t}$$
$$x(t) = ln(\frac{Ce^t - 1}{Ce^t})$$

Solving for C with initial conditions

$$2 = ln(\frac{Ce^0 - 1}{Ce^0}) = ln(e - \frac{1}{C})$$
$$\frac{1}{C} = e - e^2$$
$$C = \frac{1}{e - e^2}$$

Final Analytical Solution to Equation(1)

$$x(t) = ln(e - \frac{e - e^2}{e^t})$$

$$\text{Derivative of } \frac{dx}{dt} = (\frac{1}{x(t)} - 1)\frac{1}{\sqrt{x(t)}}$$

$$f(x,t) = x^{-3/2} - x^{-1/2}$$
$$f_x(x,t) = \frac{-3x^{-5/2}}{2} + \frac{x^{-3/2}}{2}$$

$$\text{Derivative of } f(x,t) = e^{-x+1} - 1$$

$$f_x(x,t) = -e^{-x-1}$$