APPM 2350 - TNB Wars

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1 Introduction

In this project, the unorthodox escape route of an aging transport shuttle is evaluated to see if the trajectory is sufficient to escape the enemy planet, Xanadar IV, and fool their automated defense system. The data from this analysis will be returned to High Command to determine if the mission should proceed.

The unit tangent vector $\hat{\mathrm{T}}$ is defined as the vector of magnitude one unit that is parallel to the shuttle's velocity and tangent to the shuttle's position at a specific point in time. It is an indicator of the direction of the curve.

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} \tag{1}$$

The principal unit normal vector $\hat{\mathrm{N}}$ is defined as the vector of magnitude one unit that is orthogonal to the unit tangent vector. This vector points in the direction that the curve turns.

$$\hat{N}(t) = \frac{\hat{T}'(t)}{||\hat{T}'(t)||} \tag{2}$$

The unit binormal vector $\hat{\mathrm{B}}$ is defined as the vector of magnitude one unit that is orthogonal to both the unit tangent vector and the principal normal vector.

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) \tag{3}$$

The curvature κ at a given point is a measurement of how fast the curve is changing direction at that instant. It is derived from the magnitude of the rate of change of the unit tangent vector $\hat{T}(t)$ with respect to arc length s. The unit tangent vector is used to express curvature so that the speed at which an object travels along a curve does not effect the calculation of how curvy the trajectory truly is.

$$\kappa = ||\frac{d\hat{T}}{ds}|| = ||\frac{d\hat{T}/dt}{ds/dt}|| = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3} \tag{4}$$

Torsion τ is the rate of change of the curve's osculating plane. It can be calculated by the cross product of the radius and the force perpendicular to the motion of the object.

$$\tau = ||\vec{F}\,(t)|| * ||\vec{r}\,(t)|| \sin\theta \tag{5}$$

The tangential component of acceleration a_T is the component of acceleration that is parallel to the velocity, or in the direction of the tangent. It is equivalent to the derivative of speed. Tangential acceleration describes the rate of change of speed.

$$a_T = ||\vec{v}||' = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{||\vec{r}\,'(t)||} \tag{6}$$

The normal component of acceleration a_N is the component of acceleration that is perpendicular to the velocity, or in the direction of the normal. It is equivalent to the curvature times the square of the speed. Normal acceleration describes the change in direction as it is orthogonal to the object's path.

$$a_N = \kappa||\vec{v}||^2 = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||} \tag{7}$$

2 The Escape Trajectory

The proposed escape trajectory for the shuttle is defined as

$$\vec{r}\,(t) = \langle 20[\cos\frac{8\pi}{15}t]^2, 10\sin\frac{4\pi}{15}t, \frac{6}{675}t^3\rangle, 0 \leq t \leq 15 \tag{8}$$

where the position is in kilometers (km) and the time is in minutes (min). Figure 1 shows the curve of the proposed trajectory of the escape shuttle with respect to time.

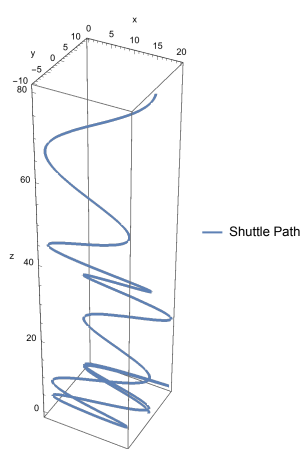

Figure 1: Shuttle Path

In order to best understand the situation that the pilot will be in, our team of commanders have completed some preliminary calculations in Mathematica to accurately analyze the proposed flight path.

We have defined the velocity $\vec{v}\,(t)$ as the derivative of the shuttle's position equation $\vec{r}\,(t)$. The instantaneous speed, then, is the magnitude of the velocity at any given point on the curve, and can be seen as a function of time in Figure 2 below.

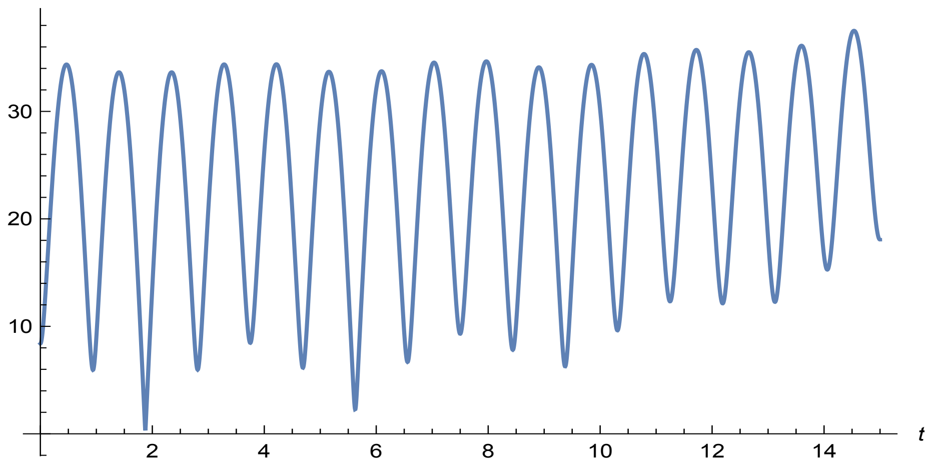

Figure 2: Instantaneous Speed vs. Time

Next, the curvature of the shuttle's path was analyzed during the time frame of its flight from 0 to 15 minutes. Curvature was calculated using equation (4). Curvature can be seen as a function of time in Figure 3 below.

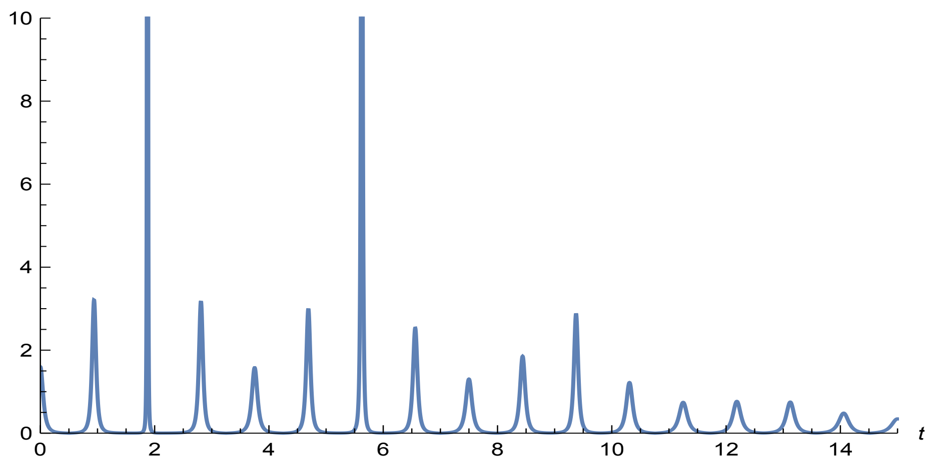

Figure 3: Curvature vs. Time

From 0 to 15 minutes, the total distance traveled by the shuttle was calculated using an arc length function in Mathematica by integrating the speed from 0 to x where x represents any time along the trajectory. In this case, the arc length was evaluated at $x = 15$ min and was calculated to be 356.558 km. The direct distance of the shuttle's path in this same time frame was measured using the magnitude of the position vector from equation (8) and was found to be $20\sqrt{17}$ km. The average speed of the shuttle over the first 15 minutes was found to be 23.7706 km/min which does exceed 20 km/min.

3 Maintaining Shuttle Integrity

The escape shuttle used in this mission is more delicate than a military spacecraft; thus, here our commanders analyze a series of safety criteria that must be met in order for the mission to be successful. The following are the safety criteria that must be satisfied for the first 15 minutes of the shuttle's trajectory:

1. The shuttle speed cannot exceed 37.5 km/min for more than 2 seconds.
2. The curvature of the trajectory cannot exceed 2000 km^{-1} at any moment in time.
3. The magnitude of both the tangential and normal components of acceleration cannot exceed 120 km/min^2 at any moment in time.

To analyze the first safety criteria, the speed of the shuttle is plotted in Figure 4 below with a horizontal line that represents 37.5 km/min. When a closer look is taken around 14 seconds, Figure 5, the speed of the shuttle does exceed 37.5 km/min. However, this only occurs for a duration of 0.0269 seconds. As this is less than 2 seconds within the first 15 minutes, this condition is satisfied.

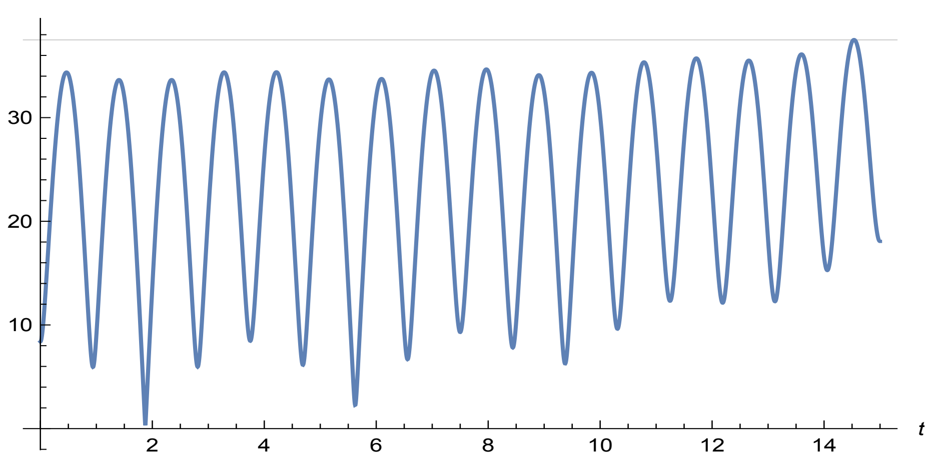

Figure 4: Shuttle Speed vs. Time; Maximum Analysis

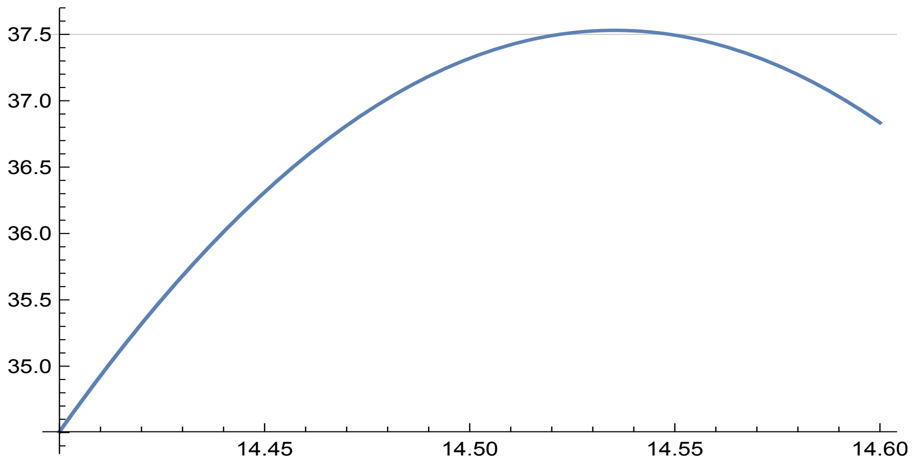

Figure 5: Shuttle Speed vs. Time; Zoomed in Maximum Analysis

To analyze the second safety criteria, the maximum curvature was found in Mathematica to be 1800.23 km^{-1}. This remains below 2000 km^{-1}; hence, the second condition is satisfied.

To analyze the third safety criteria, the tangential component of acceleration is plotted in Figure 6 and the normal component of acceleration is plotted in Figure 7 using equations (6) and (7) respectively. Both plots show a horizontal line that displays 120 $\mathrm{km/min}^2$. As both figures illustrate, the tangential and normal components of acceleration remain below the 120 $\mathrm{km/min}^2$ marker, and the third condition is satisfied.

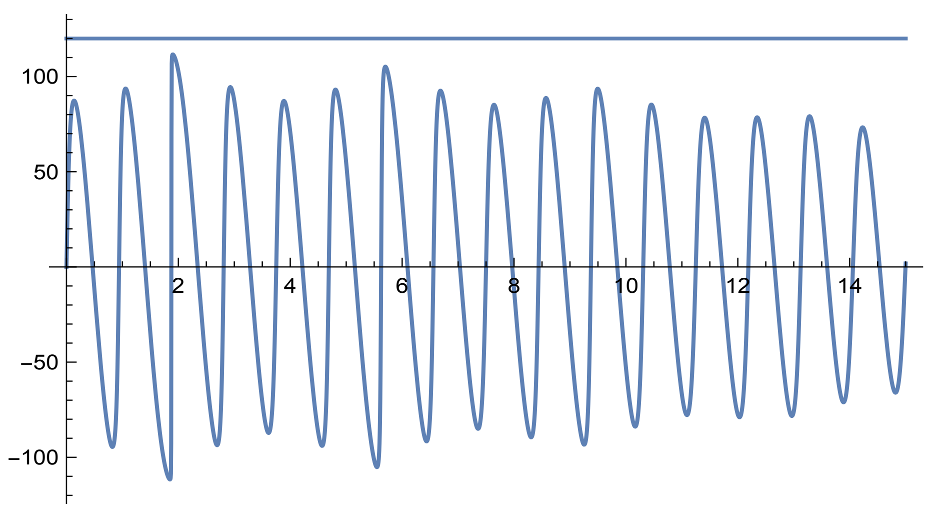

Figure 6: Tangential Acceleration

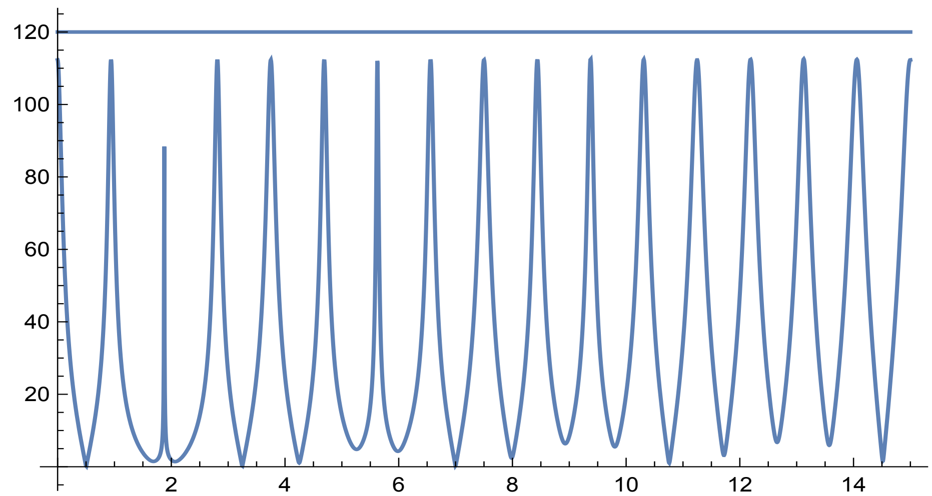

Figure 7: Normal Acceleration

From this analysis, it can be concluded that the proposed escape trajectory does meet all of the safety criteria and will, therefore, not tear the shuttle apart during its trajectory.

4 Evading the Ion Cannons

While, the escape shuttle meets all safety criteria, there are three ion cannons in the range of the shuttle that could impact the success of the mission as the shuttle is not designed to take impacts from certain directions. These cannons will begin to fire after the shuttle is 10 km above the surface of planet Xanadar IV, modelled locally as a plane. Table 1 below predicts the ion cannon locations, firing times, and constant beam velocities using data from captured enemy software.

Cannon Location (km)	Firing Time (min)	Beam Velocity (km/min)
(30,0,0)	$t = 7.5$	(-20,8,15)
(25,20,0)	$t = 8$	$(-10, -10 + \frac{5\sqrt{3}}{2}, \frac{320}{27})$
(0,-30,0)	$t = 9$	$(20,20,\frac{135}{27})$

The shuttle can survive an impact from an ion cannon as long as it does not directly collide with the shuttle in either the osculating or normal planes of the shuttle's trajectory at the time of the impact. In order to determine if the ion cannons will collide with the shuttle and the direction of the impact, the paramaterized equations for the paths taken by each ion cannon are below. Each equation takes into account the time at which the shuttle is first launched after t = 0 minutes.

$$cannon1 = \{30 - 20(-7.5 + t), 8(-7.5 + t), 15(-7.5 + t)\} \tag{9}$$

$$cannon2 = \left\{25 - 10(-8 + t), 20 + \left(-10 + \frac{5\sqrt{3}}{2}\right)(-8 + t), \frac{320}{27}(-8 + t)\right\} \tag{10}$$

$$cannon3 = \left\{20(-9 + t), -30 + 20(-9 + t), \frac{135}{32}(-9 + t)\right\} \tag{11}$$

The Reduce function in Mathematica was used to determine whether the cannons, using the paths defined above, intersect the trajectory of the shuttle by solving for a time when the position of both paths intersect at the same time, t. From this, it was confirmed that both cannon 1 and cannon 3 will not directly collide with the shuttle. However, cannon 2 will directly collide with the shuttle at time t = 10 minutes. The position of this direct collision is $(5, 5\sqrt{3}, \frac{640}{27})$.

Although cannon 1 and 2 do not directly collide with the shuttle, there is the possibility that they could intersect the path of the shuttle. Once again, the Reduce function in Mathematica was used to solve for intersections between the two paths at different times. Cannon 1 will not intersect with the shuttle's trajectory. On the other hand, cannon 2 will intersect with the shuttle's trajectory at a time of 10 minutes and at the position $(20, -10, \frac{135}{32})$. The shuttle passes through this point but at a time of 5.625 minutes.

The equations of the osculating and normal planes were identified to ensure that the shuttle could survive the impact from cannon 2. The osculating plane is formed from the unit tangent vector and the principal unit normal vector. Thus, its normal can be defined as the binormal vector. Using the point of collision and the binormal vector at time t = 10 minutes, the derived equation of the osculating plane is below.

$$0.0617x + 0.7293y + 0.6814z = 22.7758 \tag{12}$$

The normal plane is formed from the binormal vector and the prinipal unit normal vector. Thus, its normal can be defined as the unit tangent vector. Using the point of collision and the unit tangent vector at time t = 10 minutes, the derived equation of the normal plane is below.

$$-0.9619x - 0.1388y + 0.2357z = -0.4249 \tag{13}$$

To determine whether or not cannon 2 hits the escape shuttle in either of these planes, Figure 8 below was created using Mathematica. It depicts the osculating plane and the normal plane as circles, the shuttle's trajectory, the TNB frame, and the path of cannon 2. The escape shuttle's trajectory is modeled by the blue path which is identical to the path of the shuttle graphed in Figure 1. The black sphere on the path represents the shuttle at the time of the impact. The dashed red line represents the straight path of cannon 2.

The blue arrow represents the unit tangent vector at the time of the impact. The green arrow represents the principal unit normal vector at this time, and the red arrow represents the binormal vector at this time. Finally, the two circles shown represent the osculating and normal planes. The circle that is in frame with the unit tangent vector and principal unit normal vector models the osculating circle. The osculating circle can also be identified by the binormal vector orthogonal to it. The circle that is in frame with the principal unit normal and the binormal vectors models the normal circle. This normal circle can also be identified by the unit tangent vector orthogonal to it.

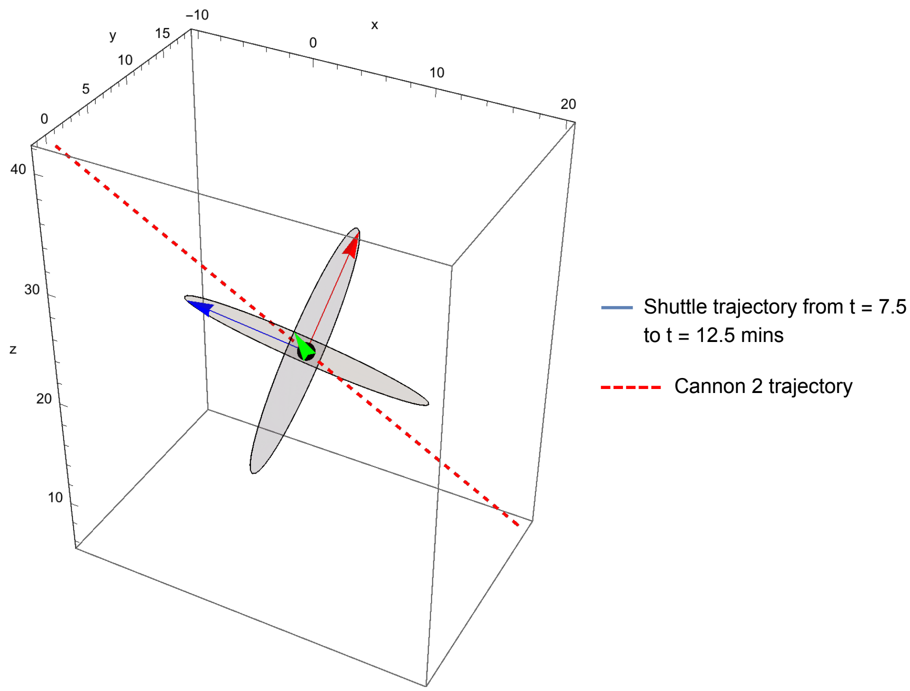

Figure 8: TNB Frames at Collision Point

5 Conclusion

Assuming the information provided by High Command is accurate, the proposed escape trajectory will allow the shuttle to survive the automated ion cannon defense system. Figure 8 illustrates the path taken by cannon 2 which does not lie within the osculating plane nor the normal plane. By substituting the parametrized equation for cannon 2 into the equations of both the osculating and normal planes, there is only one solution to t. That solution is at $t = 10$ minutes which is the time of the collision. Thus, at no other time does cannon 2 lie within either of the planes. This indicates that, although the escape shuttle will be hit at this time, it will be able to survive the impact.

Using the report above, it can be stated that the proposed plan for launching the escape shuttle and its path back to safety will be successful. Section 2 models the escape trajectory of the shuttle. Using this model, Section 3 examines the shuttle integrity and exhibits through the calculations and figures provided that the shuttle will survive the harsh conditions of space. Finally, Section 4 uncovers the potential for collisions with the enemy's ion cannons. Cannons 1 and 3 will not affect the shuttle's escape, and though cannon 1 will hit the shuttle, the shuttle will be able to survive the impact and return to safety.

If the path of the shuttle were to be parametrized differently, the analysis of this report would change. The path of the shuttle would remain the same but the shuttle would be at the points along the path at different times. Thus, the velocity and acceleration at each instantaneous point will not change but it will change for what time it is derived at. Thus, if the shuttle travels along the path at a different speed (which would create a new parameter of the function), it is possible that cannon 3 could collide with the shuttle if the parametrization changed the shuttle's trajectory so that it was at the point and time where cannon 3 intersects the path as shown in Section 4. It is also possible that a different parametrization of the path will change if the shuttle collides with cannon 2 as the shuttle could be at a different point at t = 10 minutes. Therefore, Section 2 would need to be updated to show the new parameterization. The values calculated in Section 2 (i.e. direct distance, curvature, total distance, instantaneous speed) would remain the same except these values would be at different times along the graph. Average speed in Section 2 would change as the shuttle traveling at a different speed along the curve would create a new parameterization of the function. Section 3 would give different times for maximum curvature, speed, and acceleration but the safety criteria would still be met. Section 4 would need to be re-evaluated to determine if the change in time at each location results in the evasion of a collision or a new direct collision from either cannon 2 or cannon 3.

References

1. Stewart, James. *Essential Calculus.* Second Edition, Version 2e, Brooks and Cole: Cengage Learning, 2013, pg. 591-611

2. Weisstein,Eric W."Torsion." From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/Torsion.html